ISSN: 2277-9655
Impact Factor: 4.116
CODEN: IJESS7

IJESRT
INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH TECHNOLOGY

HYBRID FIXED POINT THEOREM FOR NONLINEAR DIFFERENTIAL EQUATIONS

Sidheshwar Sangram Bellale*, Ganesh Babruwan Dapke
* Department of Mathematics, Dayanand Science College, Latur,Maharashtra, India
School of Engineering, Matoshri Pratishthan,Nanded. (M.S.), India

DOI: 10.5281/zenodo.246805

ABSTRACT
For Hybrid fixed point theorem for nonincreasing mapping in partially ordered complete metric space to prove existence as well as initial value problem of nonlinear first order ordinary differential equations.

KEYWORDS: Hybrid fixed point theorem, non-linear differential equation.

INTRODUCTION
The mixed hypothesis of algebra, topology and geometry then it is called as hybrid fixed point theorem and these hybrid fixed point theorem constitute a new stream of hybrid fixed point theory in the subject of non-linear functional analysis. It is well known that the hybrid fixed point theorem which are obtained using the mixed argument from different branches of Mathematics ,particularly the theory of non-linear differential and integral equations, (see Heikkila and Lakhsmikantham [06], Zeidler[09]and Dhage[02,03,05])

This fixed point theorem combines the metric fixed point theorem of Banach with a topological fixed point theorem of schauder in a Banach space and krannoselskiihim self applied it to some non-linear integral equations of mixed type for proving the existence results under mixed Lipschitz and compactness conditions. Recently, Ran and Recurings [08] initiated the study of hybrid fixed point theorems in partially ordered sets which is further continued in Nieto and Rodriguez –Lopez [07] and proved the hybrid fixed point theorem for the monotone.

Definition 1.1:- A partially ordered metric space $(X, \le, d)$ is called **regular** if $\{x_n\}$is a nondecreasing (respectively nonincreasing) sequence in x such that $x_n \to x^*$ as $n \to \infty$ then $x_n \le x^*$ (respectively $x_n \ge x^*$) for all $n \in N$.
Lopez and Nieto[07] gives following definition.
Condition (NL):- A partially ordered metric space X with metric d is said to satisfy condition (NL) if for every convergent sequence $\{x_n\}$ in X to the point x^* whose consecutive terms are comparable then there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that every terms is comparable to the limit x^* .
The following hybrid fixed point theorem for nonincreasing mapping is proved in Nieto and Lopez [07].
Theorem 1.1(Nieto and Rodriguez-Lopez [07]):- Let$(X, \le)$ be a partially ordered set and suppose that there is a metric d in X such that(X, d) is a complete metric space. Let $\mathrm{T}: X \to X$ be a monotone nonincreasing mapping such that there exists a constant $\mathrm{K} \in [0,1)$ such that

$$d(\mathrm{T}x, \mathrm{T}y) \le k\, d(x, y) \tag{1.1}$$

For all element $x, y \in X$, $x \ge y$. Assume that either T is continuous or X satisfies condition (NL). Further if there is an element $x_0 \in X$ satisfying $x_0 \le \mathrm{T}x_0$ or $x_0 \ge \mathrm{T}x_0$ then T has a fixed point which is further unique if ‘’every pair of element in X has a lower and an upper bound’’.

HYBRID FIXED POINT THEORY

We consider the following definition in what follows.

Condition (D):- A partially ordered metric space X with metric d is said to satisfy condition (D) if every sequence $\{x_n\}$ in X whose consecutive terms are comparable has a monotone i.e.nondecreasing or nonincreasing subsequence.

There do exist sequence in X with condition (D). For example, if we consider $X = \mathrm{R}$, then the sequence $\{x_n\}$ in R defined by $x_n = (-1)^{n+1}\frac{1}{n}$ has two subsequence, one is nondecreasing another is nonincreasing. Again, the sequence $\left\{1, \ -\frac{1}{2}, \ 3, \ -\frac{1}{4}......\right\}$ satisfies the condition (D) but not condition (NL).

Definition 2.1:- Let (X,d) be a metric space and Let $\mathrm{T}: X \to X$ be a mapping. Given an element $x \in X$, we define an orbit $O(x;\mathrm{T})$ of T at x by

$O(x:\mathrm{T}) = \{x, \ \mathrm{T}x, \ \mathrm{T}^2x,...... \ \mathrm{T}^n x......\}$

Then T is called T-orbitally continuous on X if for any sequence $\{x_n\} \subseteq O(x;\mathrm{T})$, we have that $x_n \to x^*$ $\Rightarrow Tx_n \to Tx^*$ for each $x \in X$ .The metric space X is called as $T-$orbitally complete if Cauchy sequence if $\{x_n\} \subseteq \{O,T\}$ Converges to x^* in X .

Note:-The continuity implies that T – orbitally continuity and completeness implies T – orbitally completeness of a metric space X , but converse may not be true.

Definition 2.2(Dhage [5]):- A mapping $T: X \to X$ is called partially continuous at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\| \mathrm{T}x - \mathrm{T}a \| < \epsilon$ whenever x is comparable to a and $\| x - a \| < \delta$. T Called partially continuous on X , then it is continuous on every chain C contained in X.

We frequently need a fundamental result concerning Cauchy sequence in what follows. For, we need the following definition.

Definition 2.3 (Dhage [4]):- A mapping $\Phi : R_+ \to R_+$ is called a Dominating function or, in short, function if it is an upper semi-continuous and monotonic nondecreasing function satisfying $\Phi(0) = 0$.

Lemma 2.1:- Let $\Psi : R_+ \to R_+$ be D-function satisfying $\Psi(r) < r$ for $r > 0$. the $\lim_{n\to\infty} \Psi^n(t) = 0$. for $t \in R_+$ and vice versa.

Now we are ready to state a key result in terms of D-function characterizing the Cauchy sequences in a metric space X .

Lemma 2.2:- If $\{x_n\}$ is a sequence in a metric space (X,d) satisfying

$$d(x_n, x_{n+1}) \le f(d(x_{n-1} \quad x_n)) \tag{2.1}$$

for all $n \in N$, where f is a $D-$function such that $f(r) < r, r > 0,$ then $\{x_n\}$ is Cauchy.

Theorem 2.1:- Let $(X, \le, d)$ be a partially ordered metric space. Let $\mathrm{T}: X \to X$ be a monotone nonincreasing mapping such that there exist a D-function such that

$$d(\mathrm{T}x, T^2x) \le \Phi(d(x, \quad \mathrm{T}x)) \tag{2.2}$$

For all element $x \in X$ comparable to Tx , where $\Phi(r) < r, \ r > 0$. Suppose that either X is T-orbitally complete and T is T – orbitally continuous or T is partially T – orbitally continuous and X is regular and satisfies condition (D). Further if there is an element $x_0 \in X$ satisfying $x_0 \le \mathrm{T}x_0$ or $x_0 \ge Tx_0$, then T has a fixed point x^* and the sequence $\{T^n x_0\}$ of iterations converges to x^*.

Theorem 2.2:- Let $(X, \le, d)$ be a partially ordered metric space. Let $\mathrm{T}: X \to X$ be a monotone nonincreasing mapping such that there exist a D-function such that

$$d(\mathrm{T}x, \quad \mathrm{T}y) \le \Psi(d(x, \quad y)) \tag{2.3}$$

For all comparable elements $x, y \in X$, where $\Psi(r) < r, \ r > 0.$ suppose that either X is T – orbitally complete and T is T-orbitally continuous or T is partially T-orbitally continuous and X is regular and satisfies condition (D). Further if there is an element $x_0 \in X$ satisfying $x_0 \le Tx_0$ or $x_0 \ge Tx_0$, then T Has a fixed point x^* and the sequence $\{T^n x_0\}$ of iterations converges to x^* which is further unique if "every pair of elements in X has a lower and an upper bound".

Theorem 2.3:- Let $(X, \leq, d)$ be a partially ordered metric space and Let $T: X \to X$ be monotone mapping (monotone increasing or monotone decreasing) satisfying (2.2). Suppose that either X is T-orbitally continuous or T is partially T-orbitally continuous and X is regular and satisfies condition (D). If there exist an $x_0 \in X$ with $x_0 \leq Tx_0$ or $x_0 \geq Tx_0$, then T has a fixed point x^* and the sequence $\{T^n x_0\}$ of iterations converges to x^*.

Theorem 2.4:- Let $(X, \leq, d)$ be a partially ordered complete metric space. Let $T: X \to X$ be a monotone mapping (monotone nonincreasing or monotone decreasing) satisfying (2.3). Suppose that either X is T-orbitally complete and X is regular and satisfies condition (D). If there exists an $x_0 \in X$ with $X_0 \leq Tx_0$ or $X_0 \geq Tx_0$, Then T has a fixed point x^* and the sequence $\{T^n x_0\}$ of iterations of T at x_0 converges to x^* which is further unique "every pair of elements in X has a lower and an upper bound".

APPLICATIONS TO HYBRID DIFFERENTIAL EQUATIONS

Given a closed and bounded interval $[t_0, t_{0+a}]$ of the real line R for some $t_0, a \in R$ *with* $a > 0$, Consider the initial value problem (i.e. IVP) of first order ordinary nonlinear Hybrid differential equations (In short HDE).

$$\left.\begin{aligned} x'(t) &= f(t, x(t)) + g(t, x(t)), t \in J \\ x(t_0) &= x_0 \in R \end{aligned}\right\} \tag{3.1}$$

Where $f, g: J \times R \to R$ is continuous function.

By a solution of the HDE (3.1). We mean a function $x \in C(J, R)$ that satisfies equation (1.1), where $C(J, R)$ is the space of continuous real –valued functions defined on J.

The HDE (3.1) is well-known in the literature and discussed at length for existence as well as other aspects of the solutions. The HDE (3.1) is considered in the function space $C(J, R)$ of continuous defined on J. We define a norm $\|\ .\ \|$ and the order relation $\leq$ in $C(J, R)$ by

$$\| x \| = \sup_{t \in J} |\ t\ | \tag{3.2}$$

And

$$x \leq y \Leftrightarrow x(t) \leq y(t) \tag{3.3}$$

For all $t \in J$. Clearly $C(J, R)$ in a Banach space with respect to above supremum norm and also partially ordered w.r.to the above partially order relation $\leq$ in it. It is known that the partially ordered Banach space $C(J, R)$ is regular as well as lattice.

Definition 3.1:- A function $u \in C(J, R)$ is said to be a lower solution of the HDE (1.1) if it satisfies

$$\left.\begin{aligned} u'(t) &\leq f(t, u(t)) + g(t, u(t)) \\ u(t_0) &\leq x_0 \end{aligned}\right\} \quad \text{For all } t \in J$$

We consider the following set of assumptions in what follows.

(A_1) There exist constants $\lambda > 0$ and $\mu > 0$, with $\lambda \geq \mu$ such that

$$\frac{-[\alpha(x - y + \beta(x - y))]}{1 + (x - y)} \leq [(\lambda x) + [f(t, x)] + [g(t, x)]] - [(\lambda y) + [f(t, y)] + g(t, y)] \leq 0$$

For all $t \in J$ and $x, y \in R$, $x \geq y$.

(A_2) The HDE (1.1) has a lower solution $u \in C(J, R)$.

Consider the IVP of the HDE

$$\left.\begin{aligned} x'(t)+\lambda x(t) &= \bar{f}(t,x(t))+\bar{g}(t,x(t)) \\ x(t_0) &= x_0 \end{aligned}\right\} \tag{3.4}$$

For all $t\in J$, where $\bar{f},\bar{g}:J\times R\rightarrow R$ and

$$\bar{f}(t,x)+\bar{g}(t,x)=f(t,x)+g(t,x)+\lambda x \tag{3.5}$$

Remark: - Note that the condition $\bar{f}$, $\bar{g}$ is continuous on $J\times R$ and so the associated superposition Nymetski operator (F_x) is integrable on J . Again, a function $u\in C(J,R)$ is a solution of the HDE (3.4) iff it is a position of the HDE (1.1) on J .

Lemma 3.1:- A function $u\in C(J,R)$ is a solution of the HDE (3.4) iff it is a solution of the nonlinear integral equation

$$x(t)=ce^{-\lambda t}+e^{-\lambda t}\left[\int_{t_0}^{t} e^{\lambda s}\,\bar{f}(s,x(s))ds+\int_{t_0}^{t} e^{\lambda s}\,\bar{g}(s,x(s))ds\right] \tag{3.6}$$

For all $t\in J$ where C is a real number defined by $C=x_0\,e^{t_0}$.

Theorem 3.1:- Assume that hypothesis (A_1) and (A_2) hold .then the HDE (1.1) has a unique solution x^* Defined on J and the sequence $\{x_n\}$ of successive approximations defined by

$$x_{n+1}(t)=c\,e^{-\lambda t}+e^{-\lambda t}\left[\int_{t_0}^{t} e^{\lambda s}\,\bar{f}(s,x_n(s))ds+\int_{t_0}^{t} e^{\lambda s}\,\bar{g}(s,x_n(s))ds\right] \tag{3.7}$$

Where $x_0=u$ converges to x^*.

Proof: Set $E=C(J,\ R)$ and define two operators A on E by

$$A_x(t)=c\,e^{-\lambda t}+e^{-\lambda t}\left[\int_{t_0}^{t} e^{\lambda s}\,\bar{f}(s,x(s))ds+\int_{t_0}^{t} e^{\lambda s}\,\bar{g}(s,x(s))ds\right],t\in J \tag{3.8}$$

From the continuity of the integral, it follows that A defines the map $A:E\rightarrow E$. Now by lemma (3.1) The HDE (3.1) is equivalent to the operator equation.

$$A_x(t)=x(t),\quad t\in J \tag{3.9}$$

We shall show that the operator A satisfies all the condition of theorem (2.1)

First we show that A is monotone nonincreasing on E , let $x,y\in E$ be such $x\geq y$

$$\begin{aligned} A_x(t) &= c\,e^{-\lambda t}+e^{-\lambda t}\left[\int_{t_0}^{t} e^{\lambda s}\left(\bar{f}(x,x(s))+\bar{g}(s,x(s))\right)ds\right] \\ &\leq c\,e^{-\lambda t}+e^{-\lambda t}\left[\int_{t_0}^{t} e^{\lambda s}\left(\bar{f}(x,y(s))+\bar{g}(s,y(s))\right)ds\right] \\ &\leq A_y(t) \end{aligned}$$

$A_x(t)\leq A_y(t)$ for all $t\in J$.

This shows that A is nonincreasing operator on E into E . Next, Let $x,y\in E$ be such that $x\geq y$ .then

$$|A_x(t)-A_y(t)|=\left| e^{-\lambda t}\left[\int_{t_0}^{t} e^{\lambda s}\left(\bar{f}(s,x(s))+\bar{g}(s,x(s))\right)ds\right]-\left[\int_{t_0}^{t} e^{\lambda s}\left(\bar{f}(s,x(s))+\bar{g}(s,x(s))\right)ds\right]\right|$$

By Assumption $A_{(1)}$

$$\le e^{-\lambda t}\left[\int_{t_0}^{t} e^{\lambda s}\left[\frac{\alpha+\beta(x(s)-y(s))}{1+(x(s)-y(s))}\right]\right]ds$$

$$\le e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\,\lambda\,\frac{|x(s)-y(s)|}{1+|x(s)-y(s)|}ds$$

$$\le e^{-\lambda t}\int_{t_0}^{t}\frac{d}{ds}e^{\lambda s}\frac{\|x-y\|}{1+\|x-y\|}ds$$

$$\le e^{-\lambda t}\frac{\|x-y\|}{1+\|x-y\|}\left(e^{\lambda s}\right)_{t_0}^{t}$$

$$\le e^{-\lambda t}\frac{\|x-y\|}{1+\|x-y\|}\left(e^{\lambda t}-e^{t_0\lambda}\right)$$

$$\le \frac{\|x-y\|}{1+\|x-y\|}\left(1-e^{-\lambda(t-t_0)}\right)$$

But $\lambda>0$ hence

$$\|A_x(t)-A_y(t)\|\le\frac{\|x-y\|}{1+\|x-y\|}$$

For all $t\in j$ .Taking supremum over t, we obtain

$$\|A_x(t)-A_y(t)\|\le\psi\left(\|x-y\|\right)$$

For all $x,y\in E$ with $x\ge y$, where ψ is a D-function defined by $\psi(r)=\dfrac{r}{1+r}<r\,,r>0$. Hence A Satisfies the contraction condition (2.3) on E which further implies that A is a partially continuous and consequently partially T-orbitally continuous on E.

Next, we show that u satisfies the operator $u\le Au$. By hypothesis (A_2), the HDE (1.1) has a lower solution u .Then we have

$$\left.\begin{aligned} u'(t)&\le f(t,u(t))+g(t,u(t))\\ u(t_0)&\le x_0\end{aligned}\right\} \tag{3.10}$$

For all $t\in J$, Adding $\lambda u(t)$ on both sides of the first inequality in (3.10), we obtain

$$u'(t)+\lambda u(t)\le f(t,u(t))+g(t,u(t))+\lambda u(t),t\in J \tag{3.11}$$

Again, multiplying the above inequality (3.11) by $e^{\lambda t}$

$$e^{\lambda t}u'(t)+e^{\lambda t}\lambda u(t)\le e^{\lambda t}f(t,u(t))+e^{\lambda t}\lambda u(t)+e^{\lambda t}g(t,u(t))+e^{\lambda t}\lambda u(t)$$

$$\le e^{\lambda t}\bar{f}(t,u(t))+e^{\lambda t}\bar{g}(t,u(t))\quad \text{(By Assumption } A_2\text{)}$$

$$\left(e^{\lambda t}u(t)\right)'\le e^{\lambda t}\left[\bar{f}(t,u(t))+\bar{g}(t,u(t))\right] \tag{3.12}$$

Taking integration on the both side from t_0 to t , we get w.r.to s

$$e^{\lambda t}u(t)\le\int_{t_0}^{t}e^{\lambda s}\left(\bar{f}(s,u(s))+\bar{g}(s,u(s))ds\right)+c$$

$$u(t) \le e^{-\lambda t}\int_{t_0}^{t} e^{\lambda s}\left(\bar{f}(s,u(s))+\bar{g}(s,u(s))\right)ds + c\,e^{-\lambda t} \qquad (3.13)$$

For all $t \in J$. from definition of the operator A it follows that $u(t) \le Au(t)$ for all $t \in J$.
Hence $u \le Au$. Thus A satisfies the condition of theorem (2.2) and we apply it to conclude that the operator equation $Ax = x$ has a solution. Consequently the integral equation and the HDE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}$ of successive approximation defined by
(3.7) converges to x^*. Hence the proof.

REFERENCES

[1] G.Birkhoff, Lattice theory, Amer .Math. Soc.Publ.1947.
[2] B.C. Dhage, On extension of Tarski's fixed point theorem and application, pure Appl. Math. Sci. 25 (1987), 37-42.
[3] B.C. Dhage, Some fixed point theorems for in ordered Banach Spaces and applications, Mathematics student 61(1992), 81-88.
[4] B.C.Dhage, Fixed point theorem in ordered Banach algebras and applications, Pan Amer. Math.J.9 (4) (1999), 93-102.
[5] B.C.Dhage, Hybrid Fixed point theory in partially ordered normed linear spaces and applicationsTo fractional integral equations, Differ. Equ Appl.5 (2013), 155-184.
[6] S. Heikkila and V.Lakshmikantham, Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations, Marcel Dekker inc., New York 1994.
[7] J.J.Nieto and R.Rodriguez-Lopez, Existence and Uniqueness of fixed point in partially ordered sets And applications to ordinary differential equations, Aeta Math. Sinica (English Series) 23 (2007), 2205-2212.
[8] A.C.M. Ran, M.C.R. Reurings. A fixed point theorem in partially ordered sets and some Applications to matrix equations, proc. Amer. Math. Soc.132 (2003), 1435-1443.
[9] E.Zeidler.Nonlinear Functional Analysis and its Applications: part I, Springer Verlag 1985.